

EXTENSIONS OF RUBIO DE FRANCIA'S EXTRAPOLATION THEOREM IN VARIABLE LEBESGUE SPACE AND APPLICATION

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ABSTRACT. We obtain one variant of the extrapolation theorem of Rubio de Francia for variable exponent Lebesgue spaces. As a consequence we obtain conditions guarantee boundedness of strongly singular integral operators, singular integral operators with rough kernels, fractional maximal operators related to spherical means, Bochner-Riesz operators in variable Lebesgue spaces.

1. INTRODUCTION

Given a measurable function $p : \mathbb{R}^n \rightarrow [1, \infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on \mathbb{R}^n such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to modeling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [7], [4]). The space $L^{p(\cdot)}(\mathbb{R}^n)$ have many properties in common with the standard L^p spaces. For use below we highlight

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the fact that the associate space of $L^{p(\cdot)}(\mathbb{R}^n)$ is $L^{p'(\cdot)}(\mathbb{R}^n)$, where the conjugate exponent function $p'(\cdot)$ is defined by $1/p(x) + 1/p'(x) = 1$ with $1/\infty = 0$.

Define \mathcal{P}^0 to be the set of measurable functions $p : \mathbb{R}^n \rightarrow (0, \infty)$ such that

$$p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) > 0, \quad p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty.$$

Given $p(\cdot) \in \mathcal{P}^0$, we can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n)$, where $0 < p_0 < p^-$ and $q(\cdot) = p(\cdot)/p_0$. We can define a quasi-norm on this space by

$$\|f\|_{p(\cdot)} = \| |f|^{p_0} \|_{q(\cdot)}^{1/p_0}.$$

By a weight we mean a non-negative, locally integrable function w . Given a weight w , $L^p(w)$ will denote the weighted Lebesgue space with norm

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

Central to the study of weights are the so-called A_p weights, we say $w \in A_p$ if there exists a constant C such that for every cube $Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q w(x) dx \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \leq C < \infty, \quad \text{if } 1 < p < \infty$$

and

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \operatorname{ess\,inf}_{x \in Q} w(x), \quad \text{if } p = 1.$$

Hereafter \mathcal{F} will denote a family of pairs (f, g) of non-negative, measurable functions on \mathbb{R}^n . We say that an inequality

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx$$

holds for any $(f, g) \in \mathcal{F}$ and $w \in A_q$ (for some q , $1 \leq q < \infty$), we mean that it holds for any pair in \mathcal{F} such that the left-hand side is finite, and the constant C depends only p_0 and the A_q constant of w .

Let $B(x, r)$ denote the open ball in \mathbb{R}^n of radius r and center x . By $|B(x, r)|$ we denote the n -dimensional Lebesgue measure of $B(x, r)$. The Hardy-Littlewood maximal operator M is defined on locally integrable functions f on \mathbb{R}^n by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of exponents such that Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. (For information on exponents from $\mathcal{B}(\mathbb{R}^n)$ see the monographs [4, 7]).

In [3] the authors extending the classical extrapolation method of Rubio de Francia ([14],[15],[16]) for variable exponent Lebesgue spaces and showed that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue spaces $L^{p(\cdot)}$ whenever the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

Theorem 1.1. ([2],[3]) *Given a family \mathcal{F} , suppose that for some p_0 , $0 < p_0 < \infty$, and for every weight $w \in A_1$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx, \quad (f, g) \in \mathcal{F},$$

where C_0 depends only p_0 and the A_1 constant of w . Let $p(\cdot) \in \mathcal{P}^0$ be such that $p_0 < p_-$, and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$. Then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)},$$

where the constant C is independent of the pair (f, g) .

"Of-diagonal" generalization of Theorem 1.1 is following theorem (In the classical setting the extrapolation theorem of Rubio de Francia was extended in this manner by Harboure, Macías and Segovia [12]).

Theorem 1.2. ([2],[3]) *Given a family \mathcal{F} , assume that for some p_0 and q_0 , $0 < p_0 \leq q_0 < \infty$, and every weight $w \in A_1$,*

$$\left(\int_{\mathbb{R}^n} f(x)^{q_0} w(x) dx \right)^{1/q_0} \leq C_0 \left(\int_{\mathbb{R}^n} g(x)^{p_0} w(x)^{p_0/q_0} dx \right)^{1/p_0}, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0$ such that $p_0 < p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define the function $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in \mathbb{R}^n.$$

If $(q(\cdot)/q_0)' \in \mathcal{B}(\mathbb{R}^n)$, then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{q(\cdot)} \leq C \|g\|_{p(\cdot)}.$$

By Johnson and Neugebauer [13] and by Duoandikoetxea et al. [8] have been obtained a restricted range extrapolation theorem by restricting the class of weights. We will prove analogous kind theorem for variable exponent Lebesgue spaces.

We prove following theorems.

Theorem 1.3. *Given a family \mathcal{F} , suppose that for some p_0, δ , $0 < p_0 < \infty$, $0 < \delta < 1$, and for every weight $w \in A_1$*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w^\delta(x) dx, \quad (f, g) \in \mathcal{F}.$$

Let $\delta(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$, $p_0 < p^- \leq p^+ < \frac{p_0}{1-\delta}$. Then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}.$$

Theorem 1.4. *Given a family \mathcal{F} , assume that for some p_0, q_0 and δ , $0 < p_0 \leq q_0 < \infty$, $0 < \delta < 1$ and every weight $w \in A_1$,*

$$(1.1) \quad \left(\int_{\mathbb{R}^n} f(x)^{q_0} w(x)^\delta dx \right)^{1/q_0} \leq C_0 \left(\int_{\mathbb{R}^n} g(x)^{p_0} w(x)^{\delta p_0/q_0} dx \right)^{1/p_0}, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0$ such that $p_0 < p^- \leq p^+ < \frac{p_0 q_0}{q_0 - \delta p_0}$, define the function $q(\cdot)$ by

$$(1.2) \quad \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in \mathbb{R}^n.$$

If $\delta(q(\cdot)/q_0)' \in \mathcal{B}(\mathbb{R}^n)$, then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{q(\cdot)} \leq C \|g\|_{p(\cdot)}.$$

Note that Theorem 1.3 is a particular case of Theorem 1.4 with $p_0 = q_0$.

The following theorem in case $p_0 = 2$ proved by Duoandikoetxea, *et. al.* in [8]. The case $p_0 \neq 2$ see [2].

Theorem 1.5. *Given δ , $0 < \delta < 1$, suppose that for all $w \in A_{p_0}$, $1 < p_0 < \infty$*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w^\delta(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w^\delta(x) dx, \quad (f, g) \in \mathcal{F}.$$

Then for all p , $\frac{p_0}{1+\delta(p_0-1)} < p < \frac{p_0}{1-\delta}$, and every $w^{\frac{p_0}{p_0-p(1-\delta)}} \in A_{\frac{p_0 p \delta}{p_0 - p(1-\delta)}}$

$$\int_{\mathbb{R}^n} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) dx.$$

Using the fact that $A_1 \subset A_p$ from Theorem 1.3 and Theorem 1.5, we obtain

Corollary 1.6. *Given δ , $0 < \delta < 1$, suppose that for all $w \in A_{p_0}$, $1 < p_0 < \infty$*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w^\delta(x) dx, \quad (f, g) \in \mathcal{F}.$$

Let $\delta_*(p(\cdot)/p_*)' \in \mathbb{B}(\mathbb{R}^n)$ where $\delta_* = \frac{p_0 - p_*(1-\delta)}{p_0}$ and

$$\frac{p_0}{1 + \delta(p_0 - 1)} < p_* < p^- \leq p^+ < \frac{p_0}{1 - \delta}.$$

Then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}.$$

Remark 1.7. We can restate the hypotheses of Corollary 1.6 in term $w^\delta \in A_q \cap RH_{\frac{1}{\delta}}$, where $q = \frac{p-1+\delta}{\delta}$. and RH_s is class of weights satisfying the reverse Hölder inequality: given s , $1 < s < \infty$, we say that $w \in RH_s$ if for every cube $Q \subset \mathbb{R}^n$

$$\left(\frac{1}{|Q|} \int_Q w(x)^s dx \right)^{\frac{1}{s}} \leq \frac{C}{|Q|} \int_Q w(x) dx.$$

To do so, we need to use the following equivalence: for all δ, p , $0 < \delta < 1$, $1 < p < \infty$, $w \in A_p$ if and only if $w^\delta \in A_q \cap RH_{\frac{1}{\delta}}$, this was proved by Johnson and Neugebauer [13].

2. PROOF OF THEOREM 1.4

First note that from (1.2) obtain that

$$\frac{1}{p^-} - \frac{1}{q^-} = \frac{1}{p^+} - \frac{1}{q^+} = \frac{1}{p_0} - \frac{1}{q_0}.$$

From $p_0 < p^- \leq p^+ < \frac{p_0 q_0}{q_0 - \delta p_0}$, we obtain $q_0 < q^- \leq q^+ < \frac{q_0}{1-\delta}$.

Let $X = L^{p(\cdot)/p_0}(\mathbb{R}^n)$ and $Y = L^{q(\cdot)/q_0}(\mathbb{R}^n)$.

Let $q^- < q^+$, (for the case $q^- = q^+$ see [2]) then we can rewrite the estimate $q^+ < \frac{q_0}{1-\delta}$ in the form $1 < \frac{\delta q^+}{q^+ - q_0}$.

We have

$$1 < \frac{\delta q^+}{q^+ - q_0} \leq \frac{\delta q(x)}{q(x) - q_0} = \delta \left(\frac{q(x)}{q_0} \right)', \quad x \in \mathbb{R}^n.$$

Define the following operator $\mathcal{M}h(x) = M(h^{1/\delta})^\delta$, where M is Hardy-Littlewood maximal operator. Note that operator \mathcal{M} is bounded on Y' . Indeed

$$\begin{aligned} \|\mathcal{M}\|_{Y'} &= \|(M(h^{1/\delta}))^\delta\|_{Y'} = \|M(h^{1/\delta})\|_{\delta(q(\cdot)/q_0)'}^\delta \\ &\leq C \|h^{1/\delta}\|_{\delta(q(\cdot)/q_0)'}^\delta \leq C \|h\|_{Y'}. \end{aligned}$$

Since the operator \mathcal{M} is bounded on Y' , we can define the Rubio de Francia iteration algorithm:

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h(x)}{2^k \|\mathcal{M}\|_{Y'}^k},$$

where, for $k \geq 1$, $\mathcal{M}^k = \mathcal{M} \circ \mathcal{M} \cdots \circ \mathcal{M}$ denotes k iterations of the operator \mathcal{M} , and $\mathcal{M}^0 h = |h|$.

It follows immediately from the definition that $h(x) \leq \mathcal{R}h(x)$ and $\|\mathcal{R}\|_{Y'} \leq 2\|h\|_{Y'}$.

By the sublinearity of \mathcal{M}

$$(M(\mathcal{R}h)^{1/\delta})^\delta = \mathcal{M}(\mathcal{R}h) \leq \sum_{j=0}^{\infty} \frac{\mathcal{M}^{j+1}}{2^k \|\mathcal{M}^j\|_{X'}} \leq C \cdot \mathcal{R}(h).$$

Therefore, $(\mathcal{R}h)^{1/\delta} \in A_1$.

We can now prove the desired inequality. Fix $(f, g) \in \mathcal{F}$. Since Y is a Banach function space

$$\|f\|_{q(\cdot)}^{q_0} = \|f^{q_0}\|_Y = \sup \left\{ \int_{\mathbb{R}^n} |f(x)|^{q_0} h(x) dx : h \in Y', \|h\|_{Y'} \leq 1 \right\}.$$

Since f is non-negative, we may also restrict the supremum to non-negative h . Therefore, it will suffice to fix a function h and show that

$$\int_{\mathbb{R}^n} f(x)^{q_0} h(x) dx \leq C \|g\|_{p(\cdot)}^{q_0}$$

with a constant independent of h .

We have

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^{q_0} h(x) dx &\leq \int_{\mathbb{R}^n} f(x)^{p_0} \mathcal{R}h(x) dx \\ &\leq \|f^{q_0}\|_Y \cdot \|\mathcal{R}h\|_{Y'} \\ &\leq C \|f\|_{q(\cdot)}^{q_0} \|h\|_{Y'} < \infty, \end{aligned}$$

where we have used that $h \leq \mathcal{R}h$. Since $(\mathcal{R}h)^{1/\delta} \in A_1$, by using our hypothesis (1.1) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^{q_0} h(x) dx &\leq \int_{\mathbb{R}^n} f(x)^{q_0} \mathcal{R}h(x) dx \\ &\leq C \left(\int_{\mathbb{R}^n} g(x)^{p_0} \mathcal{R}h(x)^{p_0/q_0} dx \right)^{q_0/p_0} \\ &\leq C \|g^{p_0}\|_X^{q_0/p_0} \|(\mathcal{R}h)^{p_0/q_0}\|_{X'}^{q_0/p_0} \\ &= \|g\|_{p(\cdot)}^{q_0} \|(\mathcal{R}h)^{q_0/p_0}\|_{X'}^{q_0/p_0}. \end{aligned}$$

Note that

$$\|(\mathcal{R}h)^{q_0/p_0}\|_{X'}^{q_0/p_0} = \|\mathcal{R}h\|_{Y'} \leq C \|h\|_{Y'} = C.$$

□

3. APPLICATIONS

In this section we give a number of applications of Theorems 1.3 and 1.4 and Corollary 1.6, to show that a wide variety of classical operators are bounded on the variable $L^{p(\cdot)}$ spaces.

We start to introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition.

We say that a function $p : \mathbb{R}^n \rightarrow (0, \infty)$ is locally log-Hölder continuous on \mathbb{R}^n if there exists $c_1 > 0$ such that

$$|p(x) - p(y)| \leq c_1 \frac{1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. We say that $p(\cdot)$ satisfies the log-Hölder decay condition if there exist $p_\infty \in (0, \infty)$ and a constant $c_2 > 0$ such that

$$|p(x) - p_\infty| \leq c_2 \frac{1}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that $p(\cdot)$ is globally log-Hölder continuous in \mathbb{R}^n if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

If $p : \mathbb{R}^n \rightarrow (1, \infty)$ is globally log-Hölder continuous function in \mathbb{R}^n and $p^- > 1$, then the classical boundedness theorem for the Hardy-Littlewood maximal operator can be extended to $L^{p(\cdot)}(\mathbb{R}^n)$ (see [4, 7]). This class of exponent we denote by \mathcal{P}_{\log} . For the class of exponents \mathcal{P}_{\log} we have

Corollary 3.1. *Given δ , $0 < \delta < 1$, suppose that for all $w_0 \in A_{p_0}$, $1 < p_0 < \infty$*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w_0^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w_0^\delta(x) dx, \quad (f, g) \in \mathcal{F}.$$

Let $p(\cdot) \in \mathcal{P}_{\log}$ and

$$\frac{p_0}{1 + \delta(p_0 - 1)} < p^- \leq p^+ < \frac{p_0}{1 - \delta}.$$

Then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}.$$

Proof. Fix p_* and positive small ε such that $\frac{p_0}{1 + \delta(p_0 - 1)} < p_* < p^-$ and $p^+ < p_0/(1 - \delta) - \varepsilon$. We have

$$\frac{p_0 - p_*(1 - \delta)}{p_0} (p(\cdot)/p_*)' \geq \frac{p_0 - p_*(1 - \delta)}{p_0} \left(\frac{p_0/(1 - \delta) - \varepsilon}{p_*} \right)'$$

$$= \frac{p_0 - p_*(1 - \delta)}{p_0} \cdot \frac{p_0 - \varepsilon(1 - \delta)}{p_0 - \varepsilon(1 - \delta) - (1 - \delta)p_*} > 1.$$

It is not hard to proof $\frac{p_0 - p_*(1 - \delta)}{p_0}(p(\cdot)/p_*)' \in \mathcal{P}_{\log}$. From Corollary 1.6 we obtain desired result. \square

Singular integrals with rough kernels.

We obtain boundedness of the singular integral operator with rough kernel in variable exponent Lebesgue space we need the weighted inequalities.

Theorem 3.2. [17] *Let $n \geq 2$, $1 < r < \infty$ and let $Tf(x) = p.v.K * f(x)$ be singular integral operator with "rough" kernel*

$$K(x) = h(|x|) \frac{\Omega(x)}{|x|^n},$$

where Ω is homogeneous of degree 0 on \mathbb{R}^n , $\Omega \in L^r(S^{n-1})$, where S^{n-1} denote the unit sphere in \mathbb{R}^n . Ω has average 0 on S^{n-1} , and h is a measurable function on $(0, \infty)$ satisfying

$$\int_R^{2R} |h(t)|^r dt \leq CR \quad \text{for all } R > 0.$$

Then T is bounded on $L^p(w)(\mathbb{R}^n)$,

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w},$$

in each of the following situations:

- (A) if $r' \leq p < \infty$, and $w \in A_{\frac{p}{r'}}$, or
- (B) if $1 < p \leq r, p \neq \infty$ and $w^{\frac{-1}{p-1}} \in A_{\frac{r}{r'}}$, or
- (C) if $1 < p < \infty$ and $w^{r'} \in A_p$.

for power weight was consider in [9].

The following result concerning singular integral operator with rough kernel is known.

Corollary 3.3. ([3]) *Let $p(\cdot) \in \mathcal{P}_{\log}$, $1 < r < \infty$ and $r' < p^-$. Then singular integral operator with rough kernel T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $r' < p_0 < p^-$. As $A_1 \subset A_{\frac{p_0}{r'}}$, by Theorem 3.2(A)

$$\|Tf\|_{p_0,w} \leq C\|f\|_{p_0,w}, \quad \text{for every } w \in A_1.$$

Using Theorem 1.1 for $(|Tf|, |f|)$, we obtain that

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

\square

Corollary 3.4. *Let $p(\cdot) \in \mathcal{P}_{\log}$, $1 < r < \infty$ and $p^+ < r$. Then singular integral operator with rough kernel T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $1 < p_0 < p^-$. By Theorem 3.2(B)

$$\|Tf\|_{p_0, w} \leq C\|f\|_{p_0, w},$$

for every $w^{-\frac{1}{p_0-1}} \in A_{\frac{p'_0}{r'}}$. As we know $w^{-\frac{1}{p_0-1}} \in A_{\frac{p'_0}{r'}}$ if and only if $w^{\frac{r}{r-p_0}} \in A_{\frac{p_0 r-1}{r'}}$. Using $A_1 \subset A_{\frac{p_0 r-1}{r'}}$ we have

$$\int_{\mathbb{R}^n} T f(x)^{p_0} w_0^\delta(x) dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w_0^\delta(x) dx, \quad \text{for every } w_0 \in A_1,$$

where $\delta = \frac{r-p_0}{r}$.

It is easy to see that $\frac{p}{1-\delta} = r$ and $\delta(p^+/p_0)' > 1$ if $p^+ < r$. Therefore, $\delta(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ and using Theorem 1.3 for $(|Tf|, |f|)$, we obtain that

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

□

Corollary 3.5. *Let $p(\cdot) \in \mathcal{P}_{\log}$, $1 < r < \infty$ and for some p_0 , $1 < p_0 < \infty$, $\frac{p_0 r}{r+(r-1)(p_0-1)} < p^- \leq p^+ < r p_0$. Then singular integral operator with rough kernel T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $1 < p_0 < \infty$. by Theorem 3.2(C)

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)^\delta dx \leq C \int_{\mathbb{R}^n} |f(x)| w(x)^\delta dx, \quad \text{for every } w \in A_{p_0},$$

where $\delta = \frac{1}{r-1}$. Using Corollary 3.1 for $(|Tf|, |f|)$, we get

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

□

Strongly singular integrals. Let $\theta(\xi)$ be a smooth radial cut-off function $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. We will consider the multipliers

$$\widehat{T_{b,a}f}(\xi) = \theta(\xi) \frac{e^{i|\xi|^b}}{|\xi|^a} \widehat{f}(\xi),$$

where $0 < b < 1$ and $0 < a < nb/2$. Fefferman [10] proved that if $0 < a < nb/2$, then for p , such that $|1/p - 1/2| \leq a/nb$, then

$$\|T_{b,a}\|_p \leq c_p \|f\|_p.$$

The weighted extension of Fefferman's theorem was obtained by Chanillo [5]. Indeed If $\alpha = nb|1/p - 1/2|$, and $w \in A_p$, then for $1 < p < \infty$, $\alpha \leq a \leq nb/2$, and for γ , such that $\gamma = (a - \alpha)/(nb/2 - \alpha)$ we have

$$\|T_{b,a}f\|_{p,w^\gamma} \leq C_p \|f\|_{p,w^\gamma}.$$

Using the Corollary 3.1 we obtain

Corollary 3.6. *Let $p(\cdot) \in \mathcal{P}_{\log}$. Let for some $1 < p_0 < \infty$, $\alpha = nb|1/p_0 - 1/2|$, $\alpha \leq a \leq nb/2$, and $\gamma = (a - \alpha)/(nb/2 - \alpha)$, $\frac{p_0}{1+\gamma(p_0-1)} < p^- \leq p^+ < \frac{p_0}{1-\gamma}$. Then operator $T_{b,a}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Fractional maximal functions related to spherical means.

Denote by μ_t the normalized surface measure on the sphere in \mathbb{R}^n with center 0 and radius t . The maximal operator related to spherical means is given by

$$\mathcal{M}^\alpha = \sup_{t>0} |t^\alpha \mu_t * f|.$$

In paper [6] the authors investigate weighted $L^p \rightarrow L^q$ estimate for the maximal operators \mathcal{M}^α .

Theorem 3.7. ([6]) *Suppose that $n/n - 1 < p < q < n$, $n > 2$, that $\alpha = n/p - n/q$, and that $\max\{0, 1 - q/p'\} < \gamma \leq 1 - q/n$. Suppose also that w is in A_s , where*

$$s = \frac{q + 2p'\gamma - p'}{p'\gamma}.$$

Then there exists a constant C such that

$$\|\mathcal{M}^\alpha f\|_{q,w^\gamma} \leq C \|f\|_{p,w^{p\gamma/q}}.$$

From Theorem 1.4 and Theorem 3.7 we obtain

Corollary 3.8. *Let $0 < \alpha < n - 2$, $n > 2$, $p(\cdot) \in \mathcal{P}_{\log}$ and $n/(n - 1) < p^- \leq p^+ < n/(1 + \alpha)$. Define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \mathbb{R}^n.$$

Then

$$\|\mathcal{M}^\alpha f\|_{q(\cdot)} \leq C \|f\|_{p(\cdot)}.$$

Proof. Fix p_* such that $\frac{n}{n-1} < p_* < p^-$ and define q_* from equation $1/p_* - 1/q_* = \alpha/n$. Note that $n/(n - 1) < p_* < q_* < n$. From Theorem 3.7 we obtain

$$\|\mathcal{M}^\alpha f\|_{q_*,w^\gamma} \leq C \|f\|_{p_*,w^{p_*\gamma/q_*}},$$

for $w \in A_1$ and γ , where

$$\gamma = 1 - \frac{q_*}{n} = \frac{n - p_*\alpha - p_*}{n - p_*\alpha}.$$

We have

$$\gamma \left(\frac{q(x)}{q_0} \right)' = \frac{p(x)}{p(x) - p_*} \cdot \frac{n - \alpha p_*}{n} > \frac{n}{n - (1 + \alpha)p_*} \cdot \frac{n - \alpha p_*}{n} > 1.$$

It is not hard to proof that $\gamma \left(\frac{q(\cdot)}{q_0} \right)' \in \mathcal{P}_{\log}$ and $\frac{p_* q_*}{q_* - \gamma p_*} = \frac{n}{1 + \alpha}$. From Theorem 1.4 we obtain desired result. \square

Remark 3.9. In case $\alpha = 0$ \mathcal{M}^0 is the well-known Stein's spherical maximal operator. The $L^{p(\cdot)}(\mathbb{R}^n) \rightarrow L^{p(\cdot)}(\mathbb{R}^n)$ estimates for maximal operator \mathcal{M}^0 was investigate in [11].

Bochner-Riesz operators. The Bochner-Riesz operator in \mathbb{R}^n , ($n \geq 2$) are defined for $\beta > 0$, as

$$\widehat{T}_\beta^r(\xi) = \left(1 - \frac{|\xi|^2}{r^2} \right)_+^\beta \widehat{f}(\xi)$$

with $t_+ = \max(t, 0)$, and the maximal Bochner-Riesz operator is defined by

$$T_\beta^* f(x) = \sup_{r>0} |T_\beta^r f(x)|.$$

Theorem 3.10. ([1]) *If $0 < \beta < \frac{n-1}{2}$, then T_β^* is bounded on $L^2(w^{\frac{2\beta}{n-1}})$ for $w \in A_2$.*

Corollary 3.11. *Let $0 < \beta < \frac{n-1}{2}$, $p(\cdot) \in \mathcal{P}_{\log}$ and $\frac{2(n-1)}{n-1+2\beta} < p^- \leq p^+ < \frac{2(n-1)}{n-1-2\beta}$. Then T_β^* is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

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